

Generic entanglement and symmetry: permutation and translation symmetry

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We study entanglement of generic states in the Hilbert spaces that respect symmetry. By applying concentration of measure, we show that a pure state respecting permutation symmetry is typically weakly entangled and that a pure state respecting translation symmetry is as highly entangled as a generic state without symmetry. We then investigate higher moments of the distribution of the amount of entanglement over generic states for each symmetry and provide numerical evidence that generic states with symmetry are divided into two subensembles characterized by different entanglement spectrum, which is in contrast to those without symmetry that are divided into three subensembles. This indicates that imposing symmetry on generic states radically changes the entanglement spectrum.

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Introduction.— Symmetry is one of the guiding principles of studying many-body systems, where phases of matter are often characterized by symmetry of the system and a phase transition is understood as a consequence of the transition from one symmetry to another [1, 2]. A classification of matters by symmetry, however, does not explain all phenomena in many-body systems. For instance, it does not describe topological phases [3–5]. Entanglement is a complementary concept for classifying quantum many-body systems (see, e.g., Ref. [6] for a review), which characterizes topological phases [7, 8] and is also a key to understand critical systems [9–12]. Despite these importance of symmetry and entanglement in quantum many-body physics, little is known about their direct connections [13, 14], namely, how symmetry of a many-body system generally affects entanglement of states stably realizable in that system.

For studying properties of generic quantum states, an ensemble of pure states uniformly distributed in a given Hilbert space with respect to the unitarily invariant measure, called random states, are often exploited. Random states play an important role in physics from the black hole information paradox [15–18] to the foundation of quantum statistical mechanics [19–22]. They are also eigenstates of random Hamiltonians, which have been studied in the equilibration problems of subsystems [23, 24] and in random matrix theory [25]. Entanglement of random states called *generic entanglement* (see, e.g., Ref. [26]) is a key in these studies. It has been shown that random states in the whole Hilbert space are almost maximally entangled [27–31] and the distribution of the amount of generic entanglement has remarkable features in terms of entanglement spectrum [32–36], which has been studied for analysing universal properties of many-body systems including topological order [37–43].

In this letter, we introduce random states that respect a symmetry, equivalently eigenstates of random Hamiltonians with a symmetry, and investigate the properties of

their generic entanglement. We especially focus on permutation symmetry – a fundamental symmetry of indistinguishable fermions and bosons – and translation symmetry – the symmetry that defines the structure of a lattice. This will deepen the understanding of the properties typically observed in fermionic/bosonic and lattice systems from the viewpoint of entanglement.

Entanglement of permutationally invariant states also has an importance in quantum information. It was recently revealed that entanglement of indistinguishable bosons, induced purely by their permutation invariant nature, can be used in quantum information tasks [44]. Hence, it is important to find the amount of entanglement we can typically obtain from indistinguishable bosons, which is quantified by generic entanglement of states with permutation symmetry.

We first study the average of entanglement entropy over random states in the symmetric and antisymmetric subspaces for permutation symmetry, and in the (discrete) momentum subspaces for translation symmetry. We show that the average in the symmetric and antisymmetric subspaces is typically much smaller than that of fully random states. In the momentum subspaces, we show that the average of generic entanglement does not significantly differ from that of fully random states, implying that it is typically hard to distinguish translationally invariant states from fully random states by analysing only small subsystems. We then numerically investigate higher moments of the distribution of the amount of generic entanglement for each space. We provide evidence that imposing symmetry on random states leads to a significantly different structure of the distribution at higher moments, indicating that the entanglement spectrum of eigenstates of random Hamiltonians with and without symmetry has distinct features.

Generic entanglement.— We consider an n -qudit system Λ and denote by $\mathcal{H}^{(\Lambda)} = (\mathbb{C}^d)^{\otimes n}$ the corresponding Hilbert space. We divide the system into two subsystems A and \bar{A} , which contain n_A and $n_{\bar{A}}$ qudits, re-

spectively, and denote their Hilbert spaces by $\mathcal{H}^{(A)}$ and $\mathcal{H}^{(\bar{A})}$. We analyse the amount of entanglement of a pure state $|\phi\rangle$ with respect to the bipartition A and \bar{A} by the Rényi entropy of a reduced density matrix in A , $E_q^{(A)}(|\phi\rangle) = \log \text{tr}(\rho_A^q)/(1-q)$ for $q \in [0, \infty]$, where $\rho_A = \text{tr}_{\bar{A}}|\phi\rangle\langle\phi|$. The special case of $q = 1$, namely, $E_1^{(A)}(|\phi\rangle)$ is given by the entanglement entropy $E^{(A)}(|\phi\rangle)$, which is defined by the von Neumann entropy of ρ_A , $S(\rho_A)$. An entanglement spectrum of $|\phi\rangle$ in A is a distribution of the eigenvalues of ρ_A in decreasing order.

When a state is randomly chosen from \mathcal{H} with respect to the unitarily invariant measure, we explicitly denote it by $|\phi\rangle \in_{\mathcal{R}} \mathcal{H}$. In this letter, we sometimes refer to random states in the whole space $\mathcal{H}^{(\Lambda)}$ as *fully* random states for clarity. To introduce generic entanglement for a subspace, let $\mathcal{H}_G^{(X)}$ ($X = \Lambda, A, \bar{A}$) be a subspace in $\mathcal{H}^{(X)}$ labelled by G ($\dim \mathcal{H}_G^{(X)} = D_G^{(X)}$). We mean by *generic entanglement* for $\mathcal{H}_G^{(\Lambda)}$ entanglement of random states in a subspace described by $\mathcal{H}_G^{(\Lambda)}$.

The average entanglement entropy over fully random states satisfies $\langle E^{(A)} \rangle_{\mathcal{H}^{(\Lambda)}} > n_A \log d - d^{-n+2n_A-1}$ [28, 29]. This indicates that generic entanglement for $\mathcal{H}^{(\Lambda)}$ is almost maximal for large n . Higher moments of the distribution of the amount of entanglement, often called the entanglement distribution, has been also studied for fully random states [32–36]. By introducing a rescaled entanglement s for a given state $|\phi\rangle$ by $E_q^{(A)}(|\phi\rangle) = n_A \log d + \log s/(1-q)$, it was shown that the probability density function $P(s)$ for a state $|\phi\rangle \in_{\mathcal{R}} \mathcal{H}^{(\Lambda)}$ to have the amount of entanglement s has two singularities in terms of s [32–35]. This implies that random states are divided into three subensembles. Each subensemble is called a phase and has a distinctive entanglement spectrum [32–35]. These three phases contain maximally entangled, typical, and separable states. We call them as *maximally entangled*, *typical*, and *separable* phases, respectively.

Subspaces associated with symmetries.– The symmetric and antisymmetric subspaces in $\mathcal{H}^{(X)}$ ($X = \Lambda, A, \bar{A}$) for permutation symmetry \mathcal{P} are defined by $\mathcal{H}_{\mathcal{P},+}^{(X)} := \text{span}\{|\phi\rangle \in \mathcal{H}^{(X)} | u(\sigma)|\phi\rangle = |\phi\rangle, \forall \sigma \in \mathcal{P}\}$ and $\mathcal{H}_{\mathcal{P},-}^{(X)} := \text{span}\{|\phi\rangle \in \mathcal{H}^{(X)} | u(\sigma)|\phi\rangle = \text{sign}(\sigma)|\phi\rangle, \forall \sigma \in \mathcal{P}\}$, respectively, where $u(\sigma)$ is a unitary representation of σ . Note that $\mathcal{H}_{\mathcal{P},-}^{(\Lambda)}$ exists if and only if $n \leq d$. From a physical point of view, the symmetric (antisymmetric) subspace is a Hilbert space of indistinguishable bosons (fermions).

For translation symmetry, we consider a lattice of qudits. We only deal with the one dimensional line with the periodic boundary condition in this letter, but it is straightforward to generalize the results to higher dimensional lattices. For a line, the translation group \mathcal{T} is generated by only one generator T , which shifts every qudit by one site. Since T^n shall be the identity operator due to the periodic boundary condition, the Hilbert space $\mathcal{H}^{(X)}$ ($X = \Lambda, A, \bar{A}$) is decomposed into the dis-

crete momentum subspaces $\mathcal{H}^{(X)} = \bigoplus_{\theta} \mathcal{H}_{\mathcal{T},\theta}^{(X)}$, where $\theta \in \{\frac{2\pi k}{n}\}_{k=0,\dots,n-1}$, $\mathcal{H}_{\mathcal{T},\theta}^{(X)} := \text{span}\{|\phi\rangle \in \mathcal{H}^{(X)} | u(T)|\phi\rangle = e^{i\theta}|\phi\rangle\}$ and $u(T)$ is a unitary representation of T . This decomposition corresponds to a discrete version of Bloch's theorem [45].

Average of generic entanglement.– Our main tool to investigate the average of generic entanglement for a Hilbert space $\mathcal{H}_G^{(\Lambda)}$ is a certain type of the concentration of measure about a reduced density matrix. Since the parameter space of pure states is a hypersphere, random states are identified with random variables on a hypersphere with the uniform measure, which strongly concentrates around their average values due to the Levy's lemma [46]. By applying this argument to reduced density matrices, it was shown in Ref. [19] that for most random states reduced density matrices in the subsystem A are concentrated around a state $\Omega_G^{(A)} = \text{tr}_{\bar{A}} \Pi_G^{(\Lambda)} / D_G^{(A)}$, where $\Pi_G^{(\Lambda)}$ is a projection operator onto $\mathcal{H}_G^{(\Lambda)}$.

By combining this concentration of measure with the Fannes-Audenaert inequality [47, 48], which relates the trace distance between two states with the difference of their von Neumann entropy, we obtain the following. For $|\phi\rangle \in_{\mathcal{R}} \mathcal{H}_G^{(\Lambda)}$,

$$|E^{(A)}(|\phi\rangle) - S(\Omega_G^{(A)})| \leq \epsilon' \log R_{\phi} + \eta_0(\epsilon'), \quad (1)$$

with probability at least $1 - \exp(-D_G^{(\Lambda)} \epsilon^2 / 18\pi^3)$, where $\epsilon' = \epsilon + (\text{rank}(\Omega_G^{(A)}) / D_{\text{eff}}^{(\bar{A})})^{1/2}$, $R_{\phi} = \dim(\text{supp}(\Omega_G^{(A)}) \cup \text{supp}(\Phi^{(A)}))$, and $\eta_0(x) = -x \log x$ for $0 \leq x \leq 1/e$ and $1/e$ otherwise (see Appendix for further details). This inequality enables us to estimate the average entanglement entropy of generic entanglement for any subspace $\mathcal{H}_G^{(\Lambda)}$ by calculating the von Neumann entropy of $\Omega_G^{(A)}$.

We first investigate the average of generic entanglement for $\mathcal{H}_{\mathcal{P},\pm}^{(\Lambda)}$. Since it holds that $\Omega_{\mathcal{P},\pm}^{(A)} = \Pi_{\mathcal{P},\pm}^{(\Lambda)} / D_{\mathcal{P},\pm}^{(A)}$ and the dimension R_{ϕ} in inequality (1) is equal to $D_{\mathcal{P},\pm}^{(A)}$ for any $|\phi\rangle \in \mathcal{H}_{\mathcal{P},\pm}^{(\Lambda)}$ (see Appendix for the detailed derivation), the first moment of the distribution of entanglement entropy for the symmetric and the antisymmetric subspaces satisfies the following. For $|\phi\rangle \in_{\mathcal{R}} \mathcal{H}_{\mathcal{P},\pm}^{(\Lambda)}$,

$$|E^{(A)}(|\phi\rangle) - \log D_{\mathcal{P},\pm}^{(A)}| \leq \epsilon'_{\pm} \left(\log D_{\mathcal{P},\pm}^{(A)} - \log \epsilon'_{\pm} \right), \quad (2)$$

with probability at least $1 - \exp(-D_{\mathcal{P},\pm}^{(\Lambda)} \epsilon^2 / 18\pi^3)$, where $\epsilon'_{\pm} = \epsilon + (D_{\mathcal{P},\pm}^{(A)} / D_{\mathcal{P},\pm}^{(\bar{A})})^{1/2}$, and $D_{\mathcal{P},+}^{(X)} = \binom{n_X+d-1}{d-1}$, $D_{\mathcal{P},-}^{(X)} = \binom{d}{n_X}$ for $X = \Lambda, A, \bar{A}$ ($n_{\Lambda} = n$). The average of generic entanglement for $\mathcal{H}_{\mathcal{P},\pm}^{(\Lambda)}$ is dominantly determined by the dimension of $\mathcal{H}_{\mathcal{P},\pm}^{(\Lambda)}$. For the symmetric subspace, the average is concentrated around $E^{(A)}(|\phi\rangle) = (d-1) \log n_A$ when $n \gg n_A$ and d is constant. This is exponentially small in n_A compared to the entanglement entropy for generic entanglement of fully random states, which concentrates around $n_A \log d$. For

the antisymmetric subspace, the generic entanglement is concentrated around $\log \binom{d}{n_A}$, which is also much smaller than $n_A \log d$ if $d \geq n \gg n_A$.

These results imply that the expected amount of entanglement induced by imposing permutation symmetry on a state, such as entanglement of indistinguishable bosons and fermions, is exponentially smaller than entanglement of states in the system without symmetry. Consequently, typical properties of many-body systems investigated by using fully random states, such as the black hole paradox [15–18] and typical equilibration in subsystems [19–24], are unlikely to hold in the systems composed of indistinguishable bosons or fermions. The small amount of entanglement of states with permutation symmetry is however not surprising and is consistent with previously known results about entanglement of particular types of symmetric and antisymmetric pure states [49, 50]. In this context, our result quantitatively justifies a common belief that symmetric and antisymmetric pure states are generally weakly entangled.

For translation symmetry, we show that the average entanglement entropy of random states in the momentum subspaces does not significantly differ from that of fully random states. We first obtain that $S(\Omega_{\mathcal{T},\theta}^{(A)})$ is bounded from below by \bar{S} , which is given by

$$\bar{S} = n_A \log d - \frac{n^2}{d^{2n-3n_A}} + O\left(\frac{1}{d^{2n-3n_A}}\right), \quad (3)$$

for any $\theta \in \{\frac{2\pi k}{n}\}_{k=0,\dots,n-1}$. Since $\text{supp}(\Omega_{\mathcal{T},\theta}^{(A)}) \subset \mathcal{H}^{(A)}$ and $\text{supp}(\Phi^{(A)}) \subset \mathcal{H}^{(A)}$ hold for any $|\phi\rangle \in \mathcal{H}_{\mathcal{T},\theta}^{(A)}$, R_ϕ is bounded from above by d^{n_A} . It is also shown that $\text{rank}(\Omega_{\mathcal{T},\theta}^{(A)})/D_{\text{eff}}^{(A)} \leq n^2/d^{n_A} + O(n^2/d^{n_A})$ (see Appendix for the derivations). Combining these bounds, we obtain for $|\phi\rangle \in \mathcal{H}_{\mathcal{T},\theta}^{(A)}$ that

$$E^{(A)}(|\phi\rangle) \geq \bar{S} - \epsilon'(n_A \log d - \log \epsilon'), \quad (4)$$

with probability at least $1 - \exp(-O(\frac{d^n}{n})\epsilon'^2/18\pi^3)$, where $\epsilon' = \epsilon + nd^{-n_A/2} + o(d^{-n/2+n_A})$. Since $\bar{S} \simeq n_A \log d$ for large n and $n_A \ll n$, entanglement entropy for random states in the momentum subspaces is concentrated around $n_A \log d$ for any θ when n is sufficiently large and $\log_d n \ll n_A \ll n$. This is as high as that for fully random states (see also Fig. 1).

Unlike the average of generic entanglement for the symmetric and antisymmetric subspaces, that for the momentum subspaces is not determined by the dimension of the subspace. This is because taking a partial trace of a state in the momentum subspace does not conserve translation invariance in general, which is observed by $\text{supp}(\text{tr}_A \Pi_{\mathcal{T},\theta}^{(A)}) = \mathcal{H}^{(A)} \supsetneq \mathcal{H}_{\mathcal{T},\theta}^{(A)}$. Indeed, our result implies that the reduced density matrices of almost all states in the momentum subspaces are close to the completely mixed state in $\mathcal{H}^{(A)}$, not in $\mathcal{H}_{\mathcal{T},\theta}^{(A)}$. Thus, it is hard to distinguish random states in the momentum subspaces from fully random states by analysing the first order properties on small subsystems.

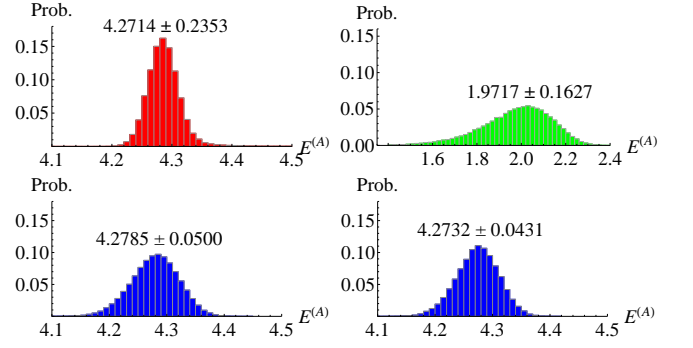


FIG. 1: The distributions of entanglement entropy $E^{(A)}$ over fully random states (left top), permutationally symmetric random states (right top), random states in $\mathcal{H}_{\mathcal{T},0}^{(A)}$ (left bottom), and those in $\mathcal{H}_{\mathcal{T},\pi}^{(A)}$ (right bottom) for $d = 2$, $n = 10$, and $n_A = 5$. The size of the samples is 10^5 , binned in intervals of 0.001 for fully random states and those in momentum subspaces, and of 0.002 for random states in the symmetric subspace. The mean and standard deviation are written in each panel.

Entanglement phases.— We numerically provide the probability density functions $P_{\text{uni}}(s)$, $P_{\mathcal{P},+}(s)$, and $P_{\mathcal{T},\theta}(s)$ of the entanglement distribution for random states in the whole Hilbert space, the symmetric and the momentum subspaces, respectively. We focus on the case for $q = 2$ since the probability density function for fully random states is well analysed in this case [32–35]. The results are given in Fig. 2.

For fully random states, two characteristic singular points of the probability density $P_{\text{uni}}(s)$ in the case of $n_A = n/2$ and $n \rightarrow \infty$ were analytically obtained in Ref. [34, 35], which we denote by $s = s_1, s_2$ and depict in Fig. 2 by dotted lines. In this case, $P_{\text{uni}}(s)$ converges to a Gaussian distribution flanked by exponentially decreasing functions on the both sides, which are connected at the transition points $s = s_1, s_2$. The three subensembles, the maximally entangled (I), typical (II), and separable (III) phases correspond to the ranges $s < s_1$, $s_1 < s < s_2$, and $s_2 < s$, respectively. In Fig. 2, the ‘phase transition’ between the maximally entangled and typical phases is clearly observed. A small difference on the transition point s_1 in the numerical result is considered to be a finite size effect [35]. The phase transition between the typical and separable phase is not clearly visible in Fig. 2. Since the finite size effect for this transition is known to be particularly large [35], this is also understood as a finite size effect.

For random states in the symmetric subspace, $P_{\mathcal{P},+}(s)$ is weakly concentrated around a much larger value of s than fully random states, which means that they are much less entangled. For random states in the momentum subspaces, the distribution concentrates around almost the same value of fully random states with a small degree of concentration, which is due to the small dimen-

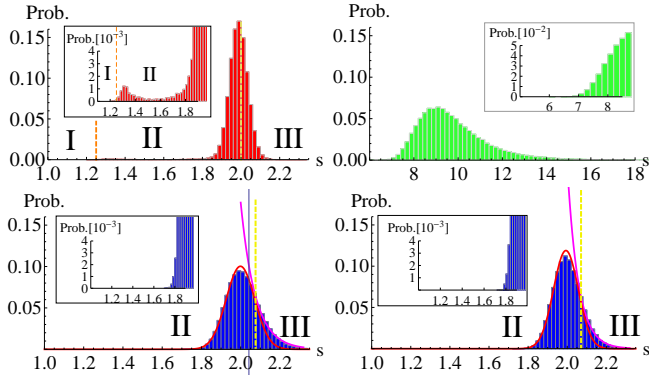


FIG. 2: The probability density functions $P(s)$ for $d = 2$, $q = 2$, $n = 10$ and $n_A = 5$ for random states in $\mathcal{H}^{(A)}$ (left top), in $\mathcal{H}_{\mathcal{P},+}^{(A)}$ (right top), in $\mathcal{H}_{\mathcal{T},0}^{(A)}$ (left bottom), and in $\mathcal{H}_{\mathcal{T},5}^{(A)}$ (right bottom). The size of the samples is 10^5 , binned in intervals of 0.001 for random states in $\mathcal{H}^{(A)}$, $\mathcal{H}_{\mathcal{T},0}^{(A)}$, and $\mathcal{H}_{\mathcal{T},5}^{(A)}$, and of 0.002 for random states in $\mathcal{H}_{\mathcal{P},+}^{(A)}$. Note that, for the rescaled entanglement s for $q > 1$, smaller s implies larger entanglement. For fully random states, the probability density is known to have two singular points at $s = s_1 = 1.25$ and $s = s_2 = 2$ when $n \rightarrow \infty$, which are indicated by vertical orange and yellow dotted lines in the figure. These points separates the ensemble into three phases I, II, and III as depicted in the panel. For random states in $\mathcal{H}_{\mathcal{T},0}^{(A)}$ and $\mathcal{H}_{\mathcal{T},5}^{(A)}$, the red lines are Gaussian distributions with $(\mu, \sigma) = (1.9988, 0.0724)$, $(1.9933, 0.0631)$, respectively, obtained by fitting the left half of the distributions. The magenta lines are exponential functions obtained by fitting the right half of the distributions. The Gaussian and exponential functions intersect at a point depicted by a vertical yellow dotted line, indicating a phase transition point.

sions of momentum subspaces. These features of $P_{\mathcal{P},+}(s)$ and $P_{\mathcal{T},\theta}(s)$ are consistent with the result on the average entanglement entropy for generic entanglement.

In contrast to the first moment of the distributions, $P_{\mathcal{P},+}(s)$ and $P_{\mathcal{T},\theta}(s)$ have features distinct from $P_{\text{uni}}(s)$ at higher moments. For both $P_{\mathcal{P},+}(s)$ and $P_{\mathcal{T},\theta}(s)$, the centre part of the distribution is described by a Gaussian distribution and the right half of the distribution exponentially decreases, however, the left tail of the Gaussian distribution is not an exponentially decreasing function. These characteristics do not depend on θ in the case of momentum subspaces. It is also observed that the transition point between the typical and separable phases for random states in the symmetric and the momentum subspaces is more clearly observed than that for fully random states. This may indicate that the finite size effect is smaller for random states in subspaces.

These results imply that imposing symmetry on the ensemble of random states significantly changes the entanglement distribution at higher moments. A phase transition between the maximally entangled and the typical phases disappears due to symmetry, suggesting that the maximally entangled phase is likely to be specific to fully random states. This is interesting particularly for

translation symmetry since the probability of obtaining the maximally entangled states is critically reduced compared to fully random states despite they have almost the same average. It is also worth mentioning that the number of states in the separable phase for random states in the momentum subspaces seems more than that of fully random states, where the separable phase is known to be physically interesting since it contains a rich structure of metastable configurations [33]. These results qualitatively indicate, in terms of the entanglement spectrum, that the spectrum for a state in the momentum subspaces is typically less concentrated around $1/d^{n_A}$. The probability for the state to have eigenvalues, which are constant in terms of n_A and are separated from the rest of the spectrum by a constant gap, becomes higher by imposing translation symmetry [26, 33].

Summary.— We have investigated generic entanglement for subspaces associated with permutation and translation symmetry. For permutation symmetry, the average entanglement entropy is exponentially smaller than that of fully random states. For translation symmetry, the average does not significantly differ from that of fully random states. However, the higher moments of the entanglement distribution and the entanglement entropy are distinct from fully random states.

It is interesting to further investigate a relation between properties of entanglement and symmetry. Since our results about the entanglement spectrum is numerical, analytical investigations are desirable. To this end, it is important to establish a general framework for the entanglement spectrum for random states in a subspace that respects a given symmetry. It will be also important from a physical point of view to investigate other symmetries related to many-body systems, such as a rotational symmetry in the Heisenberg models.

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Appendix A: Concentration of measure

We briefly explain a certain type of concentration of measure shown in Ref. [19]. We combine it with the Fannes-Audenaert inequality [47, 48] and provide a proposition that enables us to estimate the average entanglement entropy over random states in a subspace.

Let $\Pi_G^{(A)}$ be a projection operator onto $\mathcal{H}_G^{(A)}$ and define a state $\Omega_G^{(A)} = \text{tr}_{\bar{A}} \Pi_G^{(A)} / D_G^{(A)}$.

Theorem 1 (Ref. [19]) For $|\phi\rangle \in \mathcal{H}_G^{(A)}$ and $\forall \epsilon > 0$, the distance between the reduced density matrix in the subsystem A , $\Phi^{(A)} = \text{tr}_{\bar{A}} |\phi\rangle\langle\phi|$, and $\Omega_G^{(A)}$ is given prob-

abolistically by

$$\text{Prob}[\|\Phi^{(A)} - \Omega_G^{(A)}\|_1 \geq \epsilon'] \leq \exp(-\frac{D_G^{(\Lambda)} \epsilon^2}{18\pi^3}),$$

where $\|A\|_1 = \text{tr}|A|$ is the trace norm,

$$\begin{aligned} \epsilon' &= \epsilon + \sqrt{\frac{\text{rank}(\Omega_G^{(A)})}{D_{\text{eff}}^{(\bar{A})}}}, \\ D_{\text{eff}}^{(\bar{A})} &= [\text{tr}\left(\text{tr}_A \frac{\Pi_G^{(\Lambda)}}{D_G^{(\Lambda)}}\right)^2]^{-1}. \end{aligned}$$

The $D_{\text{eff}}^{(\bar{A})}$ is understood as an effective size of the subsystem \bar{A} for the state $\Pi_G^{(\Lambda)}/D_G^{(\Lambda)}$.

The Fannes-Audenaert inequality is the following;

Theorem 2 (Fannes-Audenaert inequality [47, 48]) For any two states ρ and σ on a Hilbert space \mathcal{H} with dimension D , it holds that

$$|S(\rho) - S(\sigma)| \leq \|\rho - \sigma\|_1 \log D + \eta_0(\|\rho - \sigma\|_1),$$

where

$$\eta_0(x) = \begin{cases} -x \log x & 0 \leq x \leq 1/e, \\ 1/e & 1/e < x. \end{cases}$$

Theorem 1 implies that $\|\Phi^{(A)} - \Omega_G^{(A)}\|_1 \leq \epsilon'$ with probability at least $1 - \exp(-D_G^{(\Lambda)} \epsilon^2 / 18\pi^3)$. By substituting this into the Fannes-Audenaert inequality and recalling that $\eta_0(x)$ is a monotonically increasing function and the dimension D in Theorem 2 is given by $\dim(\text{supp}(\Omega_G^{(A)} \cup \text{supp}(\Phi^{(A)})))$, we obtain the following proposition.

Proposition 1 For $|\phi\rangle \in_{\mathbb{R}} \mathcal{H}_G^{(\Lambda)}$ and $\forall \epsilon > 0$, it holds that

$$|E^{(A)}(|\phi\rangle) - S(\Omega_G^{(A)})| \leq \epsilon' \log R_\phi + \eta_0(\epsilon'),$$

with probability at least $1 - \exp(-D_G^{(\Lambda)} \epsilon^2 / 18\pi^3)$, where ϵ' is defined in Theorem 1 and $R_\phi = \dim(\text{supp}(\Omega_G^{(A)} \cup \text{supp}(\Phi^{(A)})))$.

Appendix B: Average entanglement entropy for $\mathcal{H}_{\mathcal{P},\pm}^{(\Lambda)}$

We investigate the average entanglement entropy for random states in $\mathcal{H}_{\mathcal{P},\pm}^{(\Lambda)}$ by using Proposition 1. To this end, we show $\Omega_{\mathcal{P},\pm}^{(A)} = \Pi_{\mathcal{P},\pm}^{(A)}/D_{\mathcal{P},\pm}^{(A)}$ in the following. We start with a fact that for any $\sigma_A \in \mathcal{P}$,

$$u(\sigma_A)\Omega_{\mathcal{P},\pm}^{(A)} = \Omega_{\mathcal{P},\pm}^{(A)}u(\sigma_A)^\dagger = \text{sign}(\sigma_A)^{\frac{1\mp 1}{2}}\Omega_{\mathcal{P},\pm}^{(A)}$$

holds and it implies $\text{supp}(\Omega_{\mathcal{P},\pm}^{(A)}) = \mathcal{H}_{\mathcal{P},\pm}^{(A)}$. Since we can transform $V^{\otimes n_A}\Omega_{\mathcal{P},\pm}^{(A)}(V^\dagger)^{\otimes n_A}$ for any unitary V acting on a single qudit such as

$$\begin{aligned} &V^{\otimes n_A}\Omega_{\mathcal{P},\pm}^{(A)}(V^\dagger)^{\otimes n_A} \\ &= \text{tr}_{\bar{A}}\left(V^{\otimes n_A} \otimes \text{id}_{\bar{A}}\right) \frac{\Pi_{\mathcal{P},\pm}^{(\Lambda)}}{D_{\mathcal{P},\pm}^{(\Lambda)}} \left((V^\dagger)^{\otimes n_A} \otimes \text{id}_{\bar{A}}\right) \\ &= \text{tr}_{\bar{A}} V^{\otimes n} \frac{\Pi_{\mathcal{P},\pm}^{(\Lambda)}}{D_{\mathcal{P},\pm}^{(\Lambda)}} (V^\dagger)^{\otimes n} = \text{tr}_{\bar{A}} \frac{\Pi_{\mathcal{P},\pm}^{(\Lambda)}}{D_{\mathcal{P},\pm}^{(\Lambda)}} = \Omega_{\mathcal{P},\pm}^{(A)}, \end{aligned}$$

$\Omega_{\mathcal{P},\pm}^{(A)}$ commutes with $V^{\otimes n_A}$. These implies, due to Schur's lemma [51], $\Omega_{\mathcal{P},\pm}^{(A)}$ is proportional to $\Pi_{\mathcal{P},\pm}^{(A)}$. Recalling that $\Omega_{\mathcal{P},\pm}^{(A)}$ is a state, we obtain $\Omega_{\mathcal{P},\pm}^{(A)} = \Pi_{\mathcal{P},\pm}^{(A)}/D_{\mathcal{P},\pm}^{(A)}$.

Since this immediately implies that the dimension R_ϕ in Proposition 1 is given by $D_{\mathcal{P},\pm}^{(A)}$ for any $|\phi\rangle \in \mathcal{H}_{\mathcal{P},\pm}^{(\Lambda)}$, it follows from Proposition 1 that

Proposition 2 For $|\phi\rangle \in_{\mathbb{R}} \mathcal{H}_{\mathcal{P},\pm}^{(\Lambda)}$,

$$|E^{(A)}(|\phi\rangle) - \log D_{\mathcal{P},\pm}^{(A)}| \leq \epsilon'_\pm \left(\log D_{\mathcal{P},\pm}^{(A)} - \log \epsilon'_\pm\right),$$

with probability at least $1 - \exp(-D_{\mathcal{P},\pm}^{(\Lambda)} \epsilon^2 / 18\pi^3)$, where

$$\begin{aligned} \epsilon'_\pm &= \epsilon + \sqrt{\frac{D_{\mathcal{P},\pm}^{(\Lambda)}}{D_{\mathcal{P},\pm}^{(\bar{A})}}}, \\ D_{\mathcal{P},+}^{(X)} &= \binom{n_X + d - 1}{d - 1}, \\ D_{\mathcal{P},-}^{(X)} &= \binom{d}{n_X}, \end{aligned}$$

for $X = \Lambda, A, \bar{A}$ ($n_\Lambda = n$).

Appendix C: Average entanglement entropy for $\mathcal{H}_{\mathcal{T},\theta}^{(\Lambda)}$

We show the following proposition about the average entanglement entropy for random states in the momentum subspaces $\mathcal{H}_{\mathcal{T},\theta}^{(\Lambda)}$.

Proposition 3 For $|\phi\rangle \in_{\mathbb{R}} \mathcal{H}_{\mathcal{T},\theta}^{(\Lambda)}$,

$$E^{(A)}(|\phi\rangle) \geq \bar{S} - \epsilon'(n_A \log d - \log \epsilon'),$$

with probability at least $1 - \exp(-O(\frac{d^n}{n})\epsilon^2 / 18\pi^3)$, where

$$\begin{aligned} \bar{S} &= n_A \log d - \frac{n^2}{d^{2n-3n_A}} + O\left(\frac{1}{d^{2n-3n_A}}\right), \\ \epsilon' &= \epsilon + nd^{-n_A/2} + O(d^{-n/2+n_A}). \end{aligned}$$

for any θ .

θ	$D_{\mathcal{T},\theta}^{(\Lambda)}$	$\mathcal{B}_{\mathcal{T},\theta}^{(\Lambda)}$
0	6	$\{ 0000\rangle, 1000\rangle_0, 1100\rangle_0, 1010\rangle_0, 1110\rangle_0, 1111\rangle\}$
$\pi/2$	3	$\{ 1000\rangle_{\pi/2}, 1100\rangle_{\pi/2}, 1110\rangle_{\pi/2}\}$
π	4	$\{ 1000\rangle_{\pi}, 1100\rangle_{\pi}, 1010\rangle_{\pi}, 1110\rangle_{\pi}\}$
$3\pi/2$	3	$\{ 1000\rangle_{3\pi/2}, 1100\rangle_{3\pi/2}, 1110\rangle_{3\pi/2}\}$

TABLE I: The dimensions and the bases of $\mathcal{H}_{\mathcal{T},\theta}^{(\Lambda)}$ for $n = 4$. Note that $|1010\rangle_{\pi/2}$ and $|1010\rangle_{3\pi/2}$ are not included in $\mathcal{B}_{\mathcal{T},\pi/2}^{(\Lambda)}$ and $\mathcal{B}_{\mathcal{T},3\pi/2}^{(\Lambda)}$, respectively, since they are zero vectors.

To this end, we show in the following that $S(\Omega_{\mathcal{T},\theta}^{(A)}) \leq \bar{S}$ and $\text{rank}(\Omega_{\mathcal{T},\theta}^{(A)})/D_{\text{eff}}^{(\bar{A})} \leq n^2/d^{n_A} + O(d^{-n+2n_A})$. By substituting these bounds into Proposition 1 and using a trivial upper bound of R_ϕ for any $|\phi\rangle \in \mathcal{H}_{\mathcal{T},\theta}^{(\Lambda)}$ given by d^{n_A} , Proposition 3 is obtained.

We first provide bases and dimensions of $\mathcal{H}_{\mathcal{T},\theta}^{(\Lambda)}$ in Subsection C1. We provide a lower bound of $S(\Omega_{\mathcal{T},\theta}^{(A)})$ in Subsection C2. In Subsection C3, we show that $\text{rank}(\Omega_{\mathcal{T},\theta}^{(A)})/D_{\text{eff}}^{(\bar{A})} \leq n^2/d^{n_A} + O(d^{-n+2n_A})$.

1. Bases and dimensions of $\mathcal{H}_{\mathcal{T},\theta}^{(\Lambda)}$

We first present a basis in $\mathcal{H}_{\mathcal{T},\theta}^{(\Lambda)}$ and calculate its dimension $D_{\mathcal{T},\theta}^{(\Lambda)}$. Let \mathcal{C} be a set of all configurations of an n -dit sequence, $\mathcal{C} = \{0 \cdots 00, 0 \cdots 01, \dots, d-1 \cdots d-1\}$, and $\mathcal{C}_{\mathcal{T}}$ be an equivalent class of \mathcal{C} by a translation group \mathcal{T} , $\mathcal{C}_{\mathcal{T}} := \mathcal{C}/\mathcal{T}$. We consider a set of states $\{|c\rangle_\theta\}_{c \in \mathcal{C}_{\mathcal{T}}}$ ($\theta \in \{0, \frac{2\pi}{n}, \dots, \frac{2\pi(n-1)}{n}\}$) given by

$$|c\rangle_\theta = \sqrt{\alpha_c} \sum_{k=0}^{n-1} e^{i\theta k} u(T)^k |c\rangle,$$

where T is a one-site shift operator, $u(T)$ is its unitary representation and α_c is a normalization factor depending only on the configuration c .

This set of states include zero vectors. For $c \in \{\vec{0}, \vec{1}, \dots, (d-1)\}$ where $\vec{\ell} = \ell\ell \cdots \ell$ for $\ell = 1, \dots, d-1$, $|c\rangle_\theta = |c\rangle$ for $\theta = 0$ and $|c\rangle_\theta = 0$ otherwise. Moreover, when the configuration c has a periodicity for $m < n$ in the sense that $u(T)^m |c\rangle = |c\rangle$, e.g., $|100100\rangle$ for $n = 6$ and $m = 3$, there exist θ such that $|c\rangle_\theta$ is a zero vector. By removing such zero vectors, we obtain a basis $\mathcal{B}_{\mathcal{T},\theta}^{(\Lambda)}$ in $\mathcal{H}_{\mathcal{T},\theta}^{(\Lambda)}$. The projection operator $\Pi_{\mathcal{T},\theta}^{(\Lambda)}$ is hence give by

$$\Pi_{\mathcal{T},\theta}^{(\Lambda)} = \sum_{|c\rangle_\theta \in \mathcal{B}_{\mathcal{T},\theta}^{(\Lambda)}} |c\rangle_\theta \langle c|.$$

For instance, bases in $\mathcal{H}_{\mathcal{T},\theta}^{(\Lambda)}$ for $n = 4$ are given in Table. I.

It is not simple in general to calculate $D_{\mathcal{T},\theta}^{(\Lambda)}$ and α_c since they depend on the number of periodic configurations. To avoid this complication, we only consider the

case where n is a prime number in the following. In this case, there is no configuration that satisfies $u(T)^m |c\rangle = |c\rangle$ for $m < n$, except $c \in \{\vec{0}, \vec{1}, \dots, (d-1)\}$. The basis is given by

$$\mathcal{B}_{\mathcal{T},0}^{(\Lambda)} = \{|\vec{\ell}\rangle\}_{\ell=0,1,\dots,d-1} \cup \{|c\rangle_0\}_{c \in \mathcal{C}_{\mathcal{T}} \setminus \{\vec{0}, \vec{1}, \dots, (d-1)\}},$$

and, for $\theta \neq 0$,

$$\mathcal{B}_{\mathcal{T},\theta}^{(\Lambda)} = \{|c\rangle_\theta\}_{c \in \mathcal{C}_{\mathcal{T}} \setminus \{\vec{0}, \vec{1}, \dots, (d-1)\}}.$$

For any $|c\rangle_\theta \in \mathcal{B}_{\mathcal{T},0}^{(\Lambda)}$, $\alpha_c = 1/n$. The dimension of each subspace is given by

$$D_{\mathcal{T},\theta}^{(\Lambda)} = \begin{cases} \frac{d^n - d}{n} + d & \text{for } \theta = 0, \\ \frac{d^n - d}{n} & \text{otherwise.} \end{cases}$$

2. A lower bound of $S(\Omega_{\mathcal{T},\theta}^{(A)})$

Let us denote $\Omega_{\mathcal{T},\theta}^{(A)}$ by

$$\Omega_{\mathcal{T},\theta}^{(A)} = \sum_{A, \vec{b}_A} \omega_{a_A \vec{b}_A}^\theta |a_A\rangle \langle b_A|,$$

where $a_A = a_1 \cdots a_{n_A}$ and $b_A = b_1 \cdots b_{n_A}$ ($a_i, b_i \in \{0, 1, \dots, d-1\}$ for all $i = 1, \dots, n_A$). We derive a lower bound of $S(\Omega_{\mathcal{T},\theta}^{(A)})$ by calculating $\omega_{\vec{a}_A \vec{b}_A}^\theta$ and using a relation that

$$S(\Omega_{\mathcal{T},\theta}^{(A)}) \geq -\log \text{tr}(\Omega_{\mathcal{T},\theta}^{(A)})^2. \quad (\text{C1})$$

The diagonal terms in $\Omega_{\mathcal{T},\theta}^{(A)}$ are obtained by a simple counting argument as follows;

$$\omega_{a_A a_A}^\theta = \begin{cases} \frac{d^{n_A} + m_\theta}{n D_{\mathcal{T},\theta}^{(\Lambda)}} & \text{for } \vec{a}_A = \vec{0}, \vec{1}, \dots, (d-1), \\ \frac{d^{n_A}}{n D_{\mathcal{T},\theta}^{(\Lambda)}} & \text{otherwise,} \end{cases} \quad (\text{C2})$$

where $m_\theta = n-1$ for $\theta = 0$ and $m_\theta = -n$ for $\theta \neq 0$. This difference of m_θ is due to a fact that the basis in $\mathcal{H}_{\mathcal{T},\theta}^{(\Lambda)}$ contains $|\vec{0}\rangle, \dots, |(d-1)\rangle$ only when $\theta = 0$.

The off-diagonal terms $\omega_{\vec{a}_A \vec{b}_A}^\theta$ are non-zero if and only if there exists $v_{\vec{A}} = v_1 \cdots v_{n_A}$ ($v_i \in \{0, \dots, d-1\}$ for $i = 1, \dots, n_A$) that satisfies

$$|a_A v_{\vec{A}}\rangle = u(T)^k |b_A v_{\vec{A}}\rangle \quad (\text{C3})$$

for some $k \in \{1, \dots, n-1\}$. This is because $\Omega_{\mathcal{T},\theta}^{(A)}$ is given by

$$\Omega_{\mathcal{T},\theta}^{(A)} = \frac{1}{D_{\mathcal{T},\theta}^{(\Lambda)}} \sum \text{tr}_{\vec{A}} |c\rangle_\theta \langle c|,$$

and $|c\rangle_\theta$ is a superposition of $|c\rangle$ and its shifted states. Hence $\omega_{\vec{a}_A \vec{b}_A}^\theta = 0$ if the number of i 's in a_A ($i =$

$0, \dots, d-1$) differs from that in b_A . This means that $\Omega_{\mathcal{T},\theta}^{(A)}$ is decomposed into positive operators on the Hilbert spaces spanned by states with configurations c containing m_i of i ($i = 0, \dots, d-1$);

$$\Omega_{\mathcal{T},\theta}^{(A)} = \bigoplus_{(m_0, \dots, m_{d-1})} \omega_{\mathcal{T},\theta}^{(A)}(m_0, \dots, m_{d-1}),$$

where m_i runs from 0 to n_A under the condition that $\sum_{i=0}^{d-1} m_i = n_A$. Thus $\text{tr}(\Omega_{\mathcal{T},\theta}^{(A)})^2$ is given by

$$\text{tr}(\Omega_{\mathcal{T},\theta}^{(A)})^2 = \sum_{(m_0, \dots, m_{d-1})} \text{tr} \left(\omega_{\mathcal{T},\theta}^{(A)}(m_0, \dots, m_{d-1}) \right)^2. \quad (\text{C4})$$

The dimension of $\text{supp}(\omega_{\mathcal{T},\theta}^{(A)}(m_0, \dots, m_{d-1}))$ is $M(m_0, \dots, m_{d-1})$ where

$$M(m_0, \dots, m_{d-1}) = \frac{n_A!}{m_0! \dots m_{d-1}!}.$$

We then show that the absolute value of any off-diagonal terms in $\omega_{\mathcal{T},\theta}^{(A)}(m_0, \dots, m_{d-1})$ is not greater than $1/D_{\mathcal{T},\theta}^{(\Lambda)}$ for any (m_0, \dots, m_{d-1}) . It is clear that for a fixed k , there exists at most one $v_{\bar{A}}$ that satisfies Eq. (C3) since if $v_{\bar{A}}$ and $v'_{\bar{A}}$ satisfy Eq. (C3) for the same k , we obtain $\langle v_{\bar{A}} | v'_{\bar{A}} \rangle = 1$ by taking the inner product of them, which implies $v_{\bar{A}} = v'_{\bar{A}}$. Since $k \in \{1, \dots, n-1\}$, an off-diagonal term of $\omega_{\mathcal{T},\theta}^{(A)}(m_0, \dots, m_{d-1})$ is a summation of at most $n-1$ terms, where each term has coefficient $e^{i\theta p}/(nD_{\mathcal{T},\theta}^{(\Lambda)})$ for some $p \in \{1, \dots, n-1\}$. Thus, all off-diagonal terms of $\omega_{\mathcal{T},\theta}^{(A)}(m_0, \dots, m_{d-1})$ are bounded from above by

$$\frac{1}{nD_{\mathcal{T},\theta}^{(\Lambda)}} \left| \sum_{x=1}^{n-1} \alpha_x e^{i\theta p_x} \right| \leq \frac{1}{D_{\mathcal{T},\theta}^{(\Lambda)}}, \quad (\text{C5})$$

where $\alpha_x \in \{0, 1\}$ is an indicator function that $\alpha_x = 1$ if there exists $v_{\bar{A}}$ satisfying Eq. (C3) for $k = x$ and $\alpha_x = 0$ otherwise.

By substituting the diagonal terms Eq. (C2) and the upper bounds of off-diagonal terms Eq. (C5), into Eq. (C4), an upper bound of $\text{tr}(\Omega_{\mathcal{T},\theta}^{(A)})^2$ is by

$$\begin{aligned} \text{tr}(\Omega_{\mathcal{T},\theta}^{(A)})^2 &\leq \frac{1}{d^{n_A}(1 + m_\theta d^{1-n})^2} \\ &\times \left(1 + \frac{2m_\theta}{d^{n-1}} + \frac{m_\theta^2 d + n^2 \Gamma_A}{d^{n+n_A}} \right), \quad (\text{C6}) \end{aligned}$$

where

$$\Gamma_A = \sum_{(m_0, \dots, m_{d-1})} \left[\left(\frac{n_A!}{m_0! \dots m_{d-1}!} \right)^2 - \frac{n_A!}{m_0! \dots m_{d-1}!} \right],$$

and the summation in Γ_A runs over $m_i = 0, \dots, n_A$ ($i = 1, \dots, d-1$) under the condition that $\sum_{i=0}^{d-1} m_i = n_A$. The Γ_A satisfies $d^{n_A} < \Gamma_A < d^{2n_A}$.

From Eqs. (C1) and (C6), we finally obtain a lower bound of $S(\Omega_{\mathcal{T},\theta}^{(A)})$;

$$S(\Omega_{\mathcal{T},\theta}^{(A)}) \geq n_A \log d - \frac{n^2}{d^{2n-3n_A}} + O\left(\frac{1}{d^{2n-3n_A}}\right),$$

for any θ .

3. An upper bound of $\text{rank}(\Omega_{\mathcal{T},\theta}^{(A)})/D_{\text{eff}}^{(\bar{A})}$

Due to Eq. (C6), $1/D_{\text{eff}}^{(\bar{A})} = \text{tr}(\Omega_{\mathcal{T},\theta}^{(A)})^2$ is bounded from above such as

$$\begin{aligned} 1/D_{\text{eff}}^{(\bar{A})} &\leq \frac{1}{d^{n_{\bar{A}}}(1 + m_\theta d^{1-n})^2} \left(1 + \frac{2m_\theta}{d^{n-1}} + \frac{m_\theta^2 d + n^2 \Gamma_{\bar{A}}}{d^{n+n_A}} \right) \\ &= \frac{1}{d^{n_{\bar{A}}}} \left(1 + \frac{n^2 \Gamma_{\bar{A}}}{d^{n+n_A}} + O(d^{-n-n_A}) \right). \end{aligned}$$

where we used $1 \ll n$ and $d^{n_{\bar{A}}} < \Gamma_{\bar{A}} < d^{2n_{\bar{A}}}$. Since $\text{rank}(\Omega_{\mathcal{T},\theta}^{(A)})$ is trivially bounded from above by d^{n_A} , it follows that

$$\begin{aligned} \text{rank}(\Omega_{\mathcal{T},\theta}^{(A)})/D_{\text{eff}}^{(\bar{A})} &\leq \frac{d^{n_A}}{d^{n_{\bar{A}}}} \left(1 + \frac{n^2 \Gamma_{\bar{A}}}{d^{n+n_A}} + O(d^{-n-n_A}) \right) \\ &\leq \frac{d^{n_A}}{d^{n_{\bar{A}}}} \left(1 + \frac{n^2 d^{2n_{\bar{A}}}}{d^{n+n_A}} + O(d^{-n-n_A}) \right) \\ &= \frac{n^2}{d^{n_A}} + O(d^{-n+2n_A}). \end{aligned}$$

The last equality holds for $1 \ll n^2 d^{n-3n_A}$.

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